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Ising model with a boundary magnetic field — an example of a boundary flow

Anatoly Konechny

Department of Physics and Astronomy, Rutgers, The State University of New Jersey
Piscataway, New Jersey 08854-8019 U.S.A.
E-mail: anatolyk@physics.rutgers.edu

Abstract: In [1] a nonperturbative proof of the g-theorem of Affleck and Ludwig was put forward. In this paper we illustrate how the proof of [1] works on the example of the 2D Ising model at criticality perturbed by a boundary magnetic field. For this model we present explicit computations of all the quantities entering the proof including various contact terms. A free massless boson with a boundary mass term is considered as a warm-up example.

Keywords: Renormalization Group, Field Theories in Lower Dimensions, Boundary Quantum Field Theory.

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Contents

1. Introduction

A near critical 1D quantum system with boundary can be described by a 2D quantum field theory defined on an infinite half-cylinder of circumference $\beta = 2\pi/T$ where T is the system's temperature. Let $(x, \tau) : 0 \le x < \infty,\ \tau \sim \tau + \beta$ be coordinates on the half-cylinder. We put the boundary at $x = 0$.

We are interested in systems critical in the bulk but not critical on the boundary. Physically the thermodynamic limit is taken by considering a system of finite length L with some boundary condition specified at $x = L$ and then sending L to infinity. In the limit $L \to \infty$ the two boundaries decouple and the leading term in the partition function is

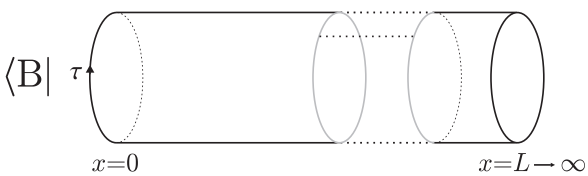

Figure 1: Coordinates on a half-cylinder.

$$Z \sim e^{\pi c L/6\beta} z z'$$

where the constants z and z' are the boundary partition functions corresponding to the boundary conditions at $x = 0$ and $x = L$ respectively. If a boundary condition is represented by a (properly normalized) boundary state $|B\rangle$ in the quantization in which the Euclidean time is along the x direction then $z = \langle B|0\rangle$ where $|0\rangle$ is the SL$(2,\mathbb{C})$-invariant conformal vacuum. The partition function Z can be also written as $\mathrm{Tr}\, e^{-\beta H_{0L}}$ where H_{0L} is a hamiltonian with respect to the Euclidean time chosen along the τ direction and the trace is taken in the corresponding Hilbert space. Assuming that this hamiltonian is hermitean Z is a nonnegative quantity. Thus for all pairs of boundary conditions the products zz' are nonnegartive. Since we are free to multiply all boundary states by a common phase factor all z's can be chosen to be nonnegative.[1]

We further note that the universal infinite factor $e^{\pi cL/6\beta}$ and the factor z' drop out from normalized correlators of operators inserted in the bulk and/or on the $x = 0$ boundary. Equivalently one can assume that the boundary conditions on a half-cylinder at infinity correspond to the bulk CFT conformal vacuum. The bulk stress energy tensor then decreases at infinity as

$$T^{bulk}_{\mu\nu}(x,\tau) \sim e^{-4\pi x/\beta}\,, \qquad x\to\infty\,. \tag{1.1}$$

The logarithm of the partition function with this boundary condition has the form

$$\ln Z = \frac{c\pi L}{6\beta} + \ln z_L$$

and $\lim_{L\to\infty} z_L = z$. The boundary entropy is defined as

$$s = \left(1 - \beta\frac{\partial}{\partial\beta}\right)\ln z\,. \tag{1.2}$$

Since temperature is the only dimensionful parameter $s = s(\mu\beta)$ where μ is the renormalization energy scale. At a fixed point $s = \ln z = \ln g$ where g is the universal noninteger gound state degeneracy of Affleck and Ludwig [2]. It was conjectured in [2] and shown in conformal perturbation theory in [3] that s decreases from fixed point to fixed point under RG flow. In [1] a nonperturbative proof of this statement which is also known as "g-theorem" was given. The proof proceeds via proving a stronger statement - a gradient formula

$$\frac{\partial s}{\partial\lambda^a} = -g_{ab}\beta^b\,, \tag{1.3}$$

where λ^a form a complete set of boundary coupling constants, $\beta^a(\lambda)$ are the corresponding beta functions and the metric g_{ab} is

$$g_{ab}(\lambda) = \int_0^\beta \mu d\tau_1 \int_0^\beta \mu d\tau_2 \langle\phi_a(\tau_1)\phi_b(\tau_2)\rangle_c \left(1 - \cos\left[2\pi\frac{(\tau_1-\tau_2)}{\beta}\right]\right). \tag{1.4}$$

Here $\phi_a(\tau)$ are boundary operators conjugated to couplings λ^a that is

$$\frac{\partial \ln z}{\partial\lambda^a} = \int_0^\beta \mu d\tau\langle\phi_a(\tau)\rangle\,.$$

[1]This extra care in the discussion of positivity of the boundary partition function z is needed because there is no $\mathrm{Tr}\, e^{-\beta H}$ representation of z.

Note that throughout the paper we stick to the conventions of [1] in which all operators and coupling constants are dimensionless. Hence a factor of μ appears with each integration. The gradient formula (1.3), (1.4) was to a large extent inspired by works on boundary string field theory [4]–[7].

In this paper we consider two exactly solvable models: a free boson with a boundary mass term and the Ising model at criticality with a boundary magnetic field. The first model was first studied in [5] and then used in the study of tachyon condensation in string theory [8, 9]. It describes a flow from Neumann to Dirichlet boundary condition. The boundary entropy for this model was computed in [9] and its monotonic decrease was demonstrated. Because of the zero mode the boundary entropy is infinite at the UV fixed point. From this point of view this is not a completely clean example of a boundary flow. We consider it in section 3 as a warm up example, demonstrate the gradient formula and show that the way it works is in accordance with the general proof. Our main example is the Ising model with a boundary magnetic field. It is considered in section 4. A lagrangian description of this model was given in [11]. It was further studied in [12, 13]. In [12] a local magnetization was computed and in [13] a boundary state was found. The system flows from free (with respect to the spin variables) to fixed boundary condition that corresponds to boundary spins directed along the magnetic field. We show that the boundary entropy monotonically decreases along the flow between the fixed points values. Mathematically both models being gaussian are close relatives that is revealed by similar looking answers. However the Ising model besides being a more physical example also provides a better illustration of various contact terms appearing throughout the proof. These contact terms are reviewed along with the proof of the gradient formula in the next section. For the Ising model we check that the gradient formula works and demonstrate by explicit computations that it does it according to the general proof. To finish the overview of the paper let us mention that appendix A provides a discussion of a certain type of distributions supported at a point on the boundary as well as explicit computations of the ones that appear in the models considered. Appendix B contains some integrals and series used in the computations.

2. Review of the proof

In this section we are going to review the proof of g-theorem that was found in [1]. The stress-energy tensor contains a boundary piece

$$T_{\mu\nu} = T^{\rm bulk}_{\mu\nu}(x,\tau) + \delta(\mu x)\delta_{\mu\tau}\delta_{\nu\tau}\theta(\tau)\,.$$

Here $\theta(\tau)$ is a boundary operator of canonical dimension 1. The conservation equations read

$$\partial^{\mu}T^{\rm bulk}_{\mu\nu}(x,\tau) \,=\, 0\,, \tag{2.1}$$

$$\mu T^{\rm bulk}_{x\tau}(0,\tau) + \partial_{\tau}\theta(\tau) \,=\, 0\,. \tag{2.2}$$

The trace of the stress-tensor is

$$T^{\mu}_{\mu} = \Theta_{\rm bulk}(x,\tau) + \delta(\mu x)\theta(\tau)\,. \tag{2.3}$$

Since the system is critical in the bulk $\Theta_{\text{bulk}}(x,\tau) = 0$ up to contact terms. The boundary trace θ can be decomposed into a linear combination of the boundary fields ϕ_a:

$$\theta = \beta^a(\lambda)\phi_a + h(\lambda)\mathbf{1}\,.$$

The generator of dilatations is T^μ_μ so that the renormalization group equation for $\ln z$ is

$$\mu\frac{\partial \ln z}{\partial\mu} = \iint \mu^2 d\tau dx \langle \delta(\mu x)\theta(\tau) + \Theta_{\text{bulk}}(x,\tau)\rangle = \beta^a\partial_a \ln z + \mu\beta h(\lambda) \tag{2.4}$$

while for one-point functions of boundary operators we have

$$\begin{aligned}\mu\frac{\partial}{\partial\mu}\langle\phi_a(\tau_1)\rangle &= \iint \mu^2 d\tau dx \langle[\delta(\mu x)\theta(\tau) + \Theta_{\text{bulk}}(x,\tau)]\phi_a(\tau_1)\rangle_c \\ &= \beta^b\partial_b\langle\phi_a(\tau_1)\rangle + (\gamma^b_a - \delta^b_a)\langle\phi_b(\tau_1)\rangle\,,\end{aligned} \tag{2.5}$$

where γ^b_a is the anomalous dimensions matrix.

To prove the gradient formula (1.3), (1.4) we start with the expression on the left hand side

$$g_{ab}\beta^b = \int_0^\beta \mu d\tau_1 \int_0^\beta \mu d\tau \langle\phi_a(\tau_1)\theta(\tau)\rangle_c \left(1 - \cos\left[2\pi\frac{(\tau-\tau_1)}{\beta}\right]\right)\,. \tag{2.6}$$

Since we assume that the UV behavior is governed by some fixed point the singularity in the two point function $\langle\phi_a(\tau_1)\theta(\tau)\rangle_c$ cannot be stronger than $|\tau-\tau_1|^{-2}$. Therefore the integral is convergent due to the presence of the $1-\cos[2\pi(\tau-\tau_1)/\beta]$ factor. For the purposes of the proof we need to split the above expression in two terms:

$$\beta\int d\tau\langle\phi_a(\tau_1)\theta(\tau)\rangle_c \tag{2.7}$$

and

$$A_a \equiv -\beta\int d\tau\langle\phi_a(\tau_1)\theta(\tau)\rangle_c \cos\left[2\pi\frac{(\tau-\tau_1)}{\beta}\right]\,. \tag{2.8}$$

At this point the two point function has to be treated as a distribution and the integrals are finite due to the presence of contact terms (see e.g. [14, section 1.3], for a general mathematical discussion).

Integrating by parts[2] in (2.8) we obtain

$$A_a = \frac{\beta^2}{2\pi}\int \mu d\tau\langle\phi_a(\tau_1)\partial_\tau\theta(\tau)\rangle_c \sin\left[2\pi\frac{(\tau-\tau_1)}{\beta}\right]\,.$$

By conservation law (2.2)

$$A_a = -2\beta\int \mu^2 d\tau\langle\phi_a(\tau_1)T^{\text{bulk}}_{x\tau}(0,\tau)\rangle_c v^\tau(0,\tau)\,,$$

where

$$v^\tau(0,\tau) \equiv \frac{\beta}{4\pi}\sin\left[2\pi\frac{(\tau-\tau_1)}{\beta}\right]$$

[2]Since the correlation function is a distribution integration by parts is a valid operation.

is a vector field on the boundary. This boundary vector field can be extended to a conformal Killing vector field on the whole half-cylinder. That is

$$\partial_\mu v_\nu + \partial_\nu v_\mu = g_{\mu\nu}\partial_\alpha v^\alpha \,, \qquad v^x(0,\tau) = 0 \,.$$

This vector field generates a subgroup of $\mathrm{SL}(2,\mathbb{R})$-transformations that leave the boundary point $\tau = \tau_1$ intact. Its explicit form can be most easily found in complex coordinates $w = 2\pi(x + i\tau)/\beta$. The analytic component reads

$$v^\omega = \frac{2\pi}{\beta}(v^x + iv^\tau) = \frac{1}{4}(e^{\omega-\omega_1} - e^{-\omega+\omega_1})$$

and the antianalytic one is given by the complex conjugated expression. The divergence of this vector field reads

$$\partial_\alpha v^\alpha = \cos\left[2\pi\frac{(\tau-\tau_1)}{\beta}\right]\cosh\left(\frac{2\pi x}{\beta}\right).$$

The fact that $\partial_\alpha v^\alpha(0,\tau_1) = 1$ means that the vector field acts locally around the boundary point $\tau = \tau_1$ as a dilatation.[3] Integrating by parts in the balk we obtain by (2.1)

$$A_a = 2\iint \mu^2 d\tau dx \langle\phi_a(\tau_1)T^{\mathrm{bulk}}_{\mu\nu}(x,\tau)\rangle_c \partial^\mu v^\nu \,.$$

There is no boundary term from infinity because of the boundary condition (1.1). Since v^μ is a conformal Killing vector we have

$$A_a = \iint \mu^2 d\tau dx \langle\phi_a(\tau_1)\Theta_{\mathrm{bulk}}(x,\tau)\rangle_c \partial_\alpha v^\alpha \,.$$

Now note that the operator Θ_{bulk} can have only a contact term OPE with any other operator. (We are going to discuss these contact terms in more details after we finish reviewing the proof.) Since Θ_{bulk} has canonical dimension 2 and in a renormalizable QFT ϕ_a has dimension less or equal than 1 the most singular contact terms possible are of the form $\delta(x)\delta'(\tau-\tau_1)$ and $\delta'(x)\delta(\tau-\tau')$. Noting that the difference $\partial_\alpha v^\alpha - 1$ vanishes to the second order at $x = 0$, $\tau = \tau_1$ we see that contact terms make no contribution. Therefore

$$A_a = \iint \mu^2 d\tau dx \langle\phi_a(\tau_1)\Theta_{\mathrm{bulk}}(x,\tau)\rangle_c \,.$$

Plugging this back into (2.6) we obtain

$$g_{ab}\beta^b = \int \mu d\tau_1 \iint \mu^2 d\tau dx \langle[\delta(\mu x)\theta(\tau) + \Theta_{\mathrm{bulk}}(x,\tau)]\phi_a(\tau_1)\rangle_c$$

that by RG equation (2.5) can be written as

$$g_{ab}\beta^b = \int \mu d\tau_1 \mu\frac{\partial}{\partial\mu}\langle\phi_a(\tau_1)\rangle = \left(\mu\frac{\partial}{\partial\mu} - 1\right)\partial_a \ln z = \partial_a\left(\mu\frac{\partial}{\partial\mu} - 1\right)\ln z = -\partial_a s$$

that completes the proof of the gradient formula.

[3]Note that there is a one-parameter family of $\mathrm{SL}(2,\mathbb{R})$ vector fields that leave a given boundary point intact and act as a dilatation in a neighborhood of that point. The particular vector field we chose is the best approximation of local dilatation in the sense that $\partial_\alpha v^\alpha - 1 = \mathcal{O}((\tau-\tau')^2)$ that is crucial for the proof. For all other aforementioned vector fields this is just $\mathcal{O}(\tau-\tau')$.

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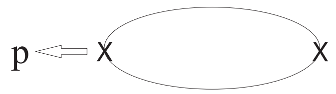

Figure 2: Feynman diagram for the two-point function of the operators $:\tilde{\phi}^2:$ and $:\phi^2:$.

In the course of proof we encountered contact terms of two kinds. The first kind has to do with nonintegrable singularities in the two point functions $\langle\phi(\tau)\theta(\tau')\rangle$. Contact terms that renormalize these nonintegrable singularities and promote the two-point functions to distributions come about when in the course of proof of the gradient formula we split expression (2.6) into 2 terms (2.7), (2.8).

A textbook example of treatment of such contact terms by standard renormalization technique arises when one considers a correlator of composite fields in free field theory

$$\langle:\tilde{\phi}^2:(\mathbf{p}):\phi^2:(0)\rangle = \int d^dx e^{-i\mathbf{p}\cdot\mathbf{x}}\langle:\phi^2:(\mathbf{x}):\phi^2:(0)\rangle$$

here $\tilde{\phi}(\mathbf{p})$ stands for the field momentum space modes. The corresponding momentum space Feynman diagram is in fig. 2. It has a divergence that is essentially of the same kind as a one-loop divergence in an interacting theory. The usual momentum space counterterms when translated into position space correspond to a linear combination of derivatives of a delta function with divergent coefficients. (See e.g. [15, section 6.2.2] for a sample computation.) Alternatively instead of working in momentum space one can employ a point splitting regularization + minimal subtraction to compute these contact terms. In the free massless boson with a boundary mass model the two point functions are integrable and there is no need for contact terms of this kind. In the Ising model with a boundary magnetic field the two point function has a nonintegrable singularity $\sim|\tau-\tau'|^{-1}$. Using the point splitting plus minimal subtraction scheme the distributional two-point function $\langle\phi(\tau)\theta(\tau')\rangle$ is computed in section 4 (see formulas (4.28), (4.29)).

Note that although such contact terms are scheme dependent the sum of terms (2.7), (2.8) is independent of the choice of contact terms.

A contact term of a different kind is a contact term in the OPE between $\Theta_{\rm bulk}(x,\tau)$ and a boundary operator $\phi(\tau')$. In a bulk theory analogous contact terms stem from the fact that $\Theta_{\rm bulk}(z,\bar{z})$ is a generator of dilatations. For instance if $\Phi(z,\bar{z})$ is a CFT primary of weights $(\Delta,\bar{\Delta})$ we have a contact term

$$\Theta_{\rm bulk}(z,\bar{z})\Phi(z',\bar{z}') = -(\Delta+\bar{\Delta})\delta^{(2)}(z-z')\Phi(z',\bar{z}')\,. \tag{2.9}$$

One can derive this contact term starting with standard OPE of $\Phi(z,\bar{z})$ with $T(z)$ and $\bar{T}(\bar{z})$ and using the Ward identity

$$2[\bar{\partial}T_{zz}+\partial T_{\bar{z}z}](z,\bar{z})\Phi(z',\bar{z}') = -\delta^{(2)}(z-z')\partial\Phi(z',\bar{z}')$$

and the complex conjugated one. Namely if we differentiate the OPE of $T(z)$ and Φ with respect to $\bar{\partial}$ and make use of the identity

$$\bar{\partial}\frac{1}{z-z'} = \pi\delta^{(2)}(z-z') \tag{2.10}$$

we obtain

$$\bar{\partial}T_{zz}(z)\Phi(z',\bar{z}') = \frac{\Delta}{2}\partial\delta^{(2)}(z-z')\Phi(z',\bar{z}') - \frac{1}{2}\delta^{(2)}(z-z')\partial\Phi(z',\bar{z}')\,.$$

Plugging this into the above Ward identity we deduce that

$$2T_{\bar{z}z}(z,\bar{z})\Phi(z',\bar{z}') = -\Delta\delta^{(2)}(z-z')\Phi(z',\bar{z}')\,.$$

A complex conjugated identity is obtained in an analogous way. Since $\Theta = 2(T_{z\bar{z}} + T_{\bar{z}z})$ we obtain (2.9).

In the presence of a boundary the generator of dilatations becomes $\Theta_{\rm bulk}(x,\tau) + \delta(x)\theta(\tau)$. The Ward identity corresponding to translations preserving the boundary reads

$$\begin{aligned}\left[\frac{i}{2}(\partial-\bar{\partial})\Theta_{\rm bulk} + 2i(\bar{\partial}T_{zz} - \partial T_{\bar{z}\bar{z}})\right]\phi(\tau') + \delta(x)[i(T_{zz}-T_{\bar{z}\bar{z}}) + \partial_{\tau}\theta]\phi(\theta') =\\ = \delta(x)\delta(\tau-\tau')\partial_{\tau'}\phi(\tau')\,.\end{aligned} \tag{2.11}$$

We see that there is a freedom in how the contact term on the right hand side can be split between the bulk and boundary[4] terms on the left hand side. In other words the contact terms between ϕ and bulk and boundary counterparts of the stress tensor are scheme dependent. Given a particular renormalization scheme we can again take OPE's of $T(z)$ and $\bar{T}(\bar{z})$ with $\phi(\tau')$ and differentiate them with respect to z and $\bar{z}$ respectively using identities similar to (2.10). Since for any $z \neq i\tau'$ the OPE's with $T(z)$ ($\bar{T}(\bar{z})$) are holomorphic (antiholomorphic) in z ($\bar{z}$) a derivative with respect to $\bar{z}$ (z) taken in a distributional sense should yield distributions supported at a boundary point τ'. A more detailed discussion of how to obtain formulas similar to (2.10) in the presence of boundary is given in appendix A. Once we have a contact term of ϕ with $\bar{\partial}T$ and $\partial\bar{T}$ we can use them in (2.11) to obtain a contact term of $\Theta_{\rm bulk}$ with ϕ. We follow this sketch of a computation for the two models we consider in the subsequent sections. Those computations are done in a particular renormalization scheme and we would like to emphasize again the scheme dependence of contact terms. Thus one could make use of a scheme in which there are no contact terms in the $\Theta_{\rm bulk}$ OPE with ϕ at all with all proper contact terms appearing in the OPE between θ and ϕ.

Note that contact terms of both kinds are not independent. They appear together in the RG equation for a one-point function of ϕ

$$\mu\frac{\partial}{\partial\mu}\langle\phi(\tau')\rangle = \int \mu d\tau\langle\phi(\tau')\theta(\tau)\rangle_c + \int\!\!\int \mu^2 dx d\tau\langle\phi(\tau')\Theta_{\rm bulk}(x,\tau)\rangle_c\,. \tag{2.12}$$

When the subtraction scheme is chosen one can compute the left hand side. If the scheme for computing the contact terms of both kinds in the correlators is chosen consistently then upon integration (2.12) should hold. Another appearance of both contact terms happens in the identity

$$\int\!\!\int \mu^2 dx d\tau\langle\phi(\tau')\Theta_{\rm bulk}(x,\tau)\rangle_c = -\int \mu d\tau\langle\phi(\tau')\theta(\tau)\rangle_c \cos\left[\frac{2\pi}{\beta}(\tau-\tau')\right] \tag{2.13}$$

that was derived in the course of proving the gradient formula.

[4]By a boundary term here we mean the term proportional to $\delta(x)$.

3. Free boson with a boundary mass term

3.1 The gradient formula

As a warm-up example we are going to consider in this section a free boson with a boundary mass term. This model was first considered in [5]. The action functional on an infinite half-cylinder reads

$$S = \frac{1}{8\pi}\int_0^\beta d\tau \int_0^\infty dx\, \partial_a\phi\partial^a\phi + \frac{u}{8\pi}\int_0^\beta \mu d\tau \phi^2\,. \tag{3.1}$$

Here μ is a renormalization scale inserted so that the boundary coupling constant u is dimensionless. Varying this action functional we obtain a boundary condition known in mathematical literature as Robin boundary condition:

$$-\frac{\partial\phi}{\partial x}(\tau,0) + u\mu\phi(\tau,0) = 0\,. \tag{3.2}$$

The field theory space for this model is labelled by a single coordinate u conjugated to a boundary operator

$$\phi_u = -\frac{1}{8\pi} :\phi^2: \,.$$

Noting that[5]

$$T_{zz} = \frac{\mu^{-2}}{4\pi} :\partial\phi\partial\phi:$$

it is easy to derive using the boundary condition (3.2)

$$(T_{zz} - T_{\bar{z}\bar{z}})|_{x=0} = -i\frac{u\mu^{-1}}{8\pi}\partial_\tau :\phi^2: \,. \tag{3.3}$$

Therefore by (2.2) the connected part of the operator θ is

$$\theta_c = -\frac{u}{8\pi}\phi^2 = \beta_u\phi_u\,,$$

where $\beta_u = u$ is the beta function of the coupling constant u.

The disk partition function was found in [5] to be

$$z = \frac{1}{2}\sqrt{\frac{\sigma}{\pi}}e^{\gamma\sigma}\Gamma(\sigma)e^{-\sigma\ln(\mu\beta/2\pi)} \tag{3.4}$$

where $\sigma = u\mu\beta/2\pi$ and Γ, γ are Euler's gamma function and constant respectively. The last factor in the above expression is absent in Witten's computations due to the specific choice of the renormalization scale μ. Also we included a normalization factor $1/2\sqrt{\pi}$ that can be computed by a method similar to the one used in [13]. From the RG equation

$$\mu\frac{\partial\ln z}{\partial\mu} = u\frac{\partial\ln z}{\partial u} + h\mu\beta$$

[5] The unusually looking factor of μ^{-2} in this equation is due to our conventions in which all operators are dimensionless.

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we find the whole operator θ:

$$\theta = \theta_c + h\mathbf{1} = -\frac{u}{8\pi} :\phi^2: -\frac{u}{2\pi}\mathbf{1}\,. \tag{3.5}$$

From (3.4) the boundary entropy is readily found to be

$$s = -\ln(2\sqrt{\pi}) + \frac{1}{2}\ln(\sigma) - \frac{1}{2} + \ln\Gamma(\sigma) - \sigma\psi(\sigma) + \sigma\,. \tag{3.6}$$

The boundary entropy (3.6) was first computed in [9] and its monotone decrease with σ was checked. In the limit $\sigma \to \infty$ the boundary entropy $s \to s_{IR} = -\ln\sqrt{2}$ that is the value corresponding to Dirichlet boundary condition.

We want to check the gradient formula

$$g_{uu}\beta_u = -\frac{\partial s}{\partial u}\,.$$

Here

$$g_{uu} = \int_0^\beta \mu d\tau_1 \int_0^\beta \mu d\tau_2 \left(1 - \cos\left[2\pi\frac{(\tau_1-\tau_2)}{\beta}\right]\right)\left\langle \frac{1}{8\pi} : \phi^2 : (\tau_1)\frac{1}{8\pi} : \phi^2 : (\tau_2)\right\rangle_c\,, \tag{3.7}$$

where the correlator is normalized and connected. The two point function was computed in [5]. On a half-cylinder it reads

$$\begin{aligned}\langle\phi(z_1)\phi(z_2)\rangle_c = &-\ln|e^{-2\pi z_1/\beta} - e^{-2\pi z_2/\beta}|^2 - \ln|1-e^{-2\pi(z_1+\bar{z}_2)/\beta}| + \\ &+\frac{2}{\sigma} - 2\sigma\sum_{n=1}^{\infty}\frac{e^{-2\pi n(z_1+\bar{z}_2)/\beta} + e^{-2\pi n(\bar{z}_1+z_2)/\beta}}{n(n+\sigma)}\,.\end{aligned} \tag{3.8}$$

The short-distance divergence in the correlator is logarithmic and thus integrable. We can then split the computation of (3.7) into computing separately the term with identity and the term with cosine. From (3.8) we obtain

$$\mu^2\int_0^\beta d\tau_1\int_0^\beta d\tau_2 \cos\left[2\pi\frac{(\tau_1-\tau_2)}{\beta}\right]\left\langle \frac{1}{8\pi} : \phi^2 : (\tau_1)\frac{1}{8\pi} : \phi^2 : (\tau_2)\right\rangle_c = \left(\frac{\mu\beta}{2\pi}\right)^2\frac{1}{\sigma}\,, \tag{3.9}$$

$$\beta\mu^2\int_0^\beta d\tau_2\left\langle \frac{1}{8\pi} : \phi^2 : (\tau_1)\frac{1}{8\pi} : \phi^2 : (\tau_2)\right\rangle_c = \left(\frac{\mu\beta}{2\pi}\right)^2\left[-\frac{1}{2\sigma^2} + \psi'(\sigma)\right]. \tag{3.10}$$

Combining (3.9) and (3.10) we obtain

$$g_{uu}\beta_u = -\left(\frac{\beta\mu}{2\pi}\right)\left[\frac{1}{2\sigma} - \sigma\psi'(\sigma) + 1\right]$$

that using (3.6) can be readily checked to coincide with $-\partial_u s$.

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3.2 Contact term in the OPE of ϕ_u with $\Theta_{\rm bulk}$

It came out in the above computation that the term (3.9) coming from integration with cosine in the metric appears on the right hand side of the gradient formula from the $\ln \mu$ term in the exponent of (3.4). This term comes from normal ordering subtraction defining the operator $:\phi^2:$. Another way to look at this logarithmic divergence is that it arises from mixing of operator $:\phi^2:$ with the identity operator under a scale transformation. Indeed according to the proof of g-theorem reviewed in section 2 the term (3.9) should reproduce the bulk part of the dilatation operator acting on ϕ_u. As it was already mentioned in section 2 this term can be fixed by the RG equation (2.12). Assuming that ϕ_u has no contact terms in its OPE with $\theta(\tau)$[6] we can compute the first term on the right hand side of (2.12), it is given by (3.10). The left hand side of (2.12) can be computed using (3.4) as

$$\mu\frac{\partial}{\partial\mu}\langle\phi_u\rangle = \mu\frac{\partial}{\partial\mu}\frac{\partial}{\partial u}\ln z\,.$$

It can be checked then that the integrated OPE with $\Theta_{\rm bulk}$ does match with the cosine term in the gradient formula that is (3.9) multiplied by a factor of u. It follows that the OPE itself is fixed assuming that it takes up the form

$$\Theta_{\rm bulk}(z,\bar z)\phi_u(\tau') = C\delta^{(2)}(z-i\tau')\mathbf{1}$$

where C is some constant. It is instructive to derive this contact term directly in a certain regularization scheme using local conservation equations. Consider the bulk-to-boundary OPE of $T_{zz}(z)$ with the boundary operator ϕ_u. We are interested in the most singular part of the short distance expansion. It can be computed using the bulk-to-boundary propagator with the Neumann boundary condition. We have

$$T_{zz} = \frac{1}{4\pi}:\partial\phi\partial\phi:$$

and the OPE

$$\begin{aligned} T_{zz}(z)\phi_u &= -\frac{1}{32\pi^2}:\partial\phi\partial\phi(z)::\phi^2(\tau'): \\ &= -\frac{1}{16\pi^2}\left(-2\frac{2\pi/\beta}{1-e^{-2\pi(z-i\tau')/\beta}}\right)^2\mathbf{1}+\cdots = \frac{1}{2\pi^2}\left(\frac{2\pi}{\beta}\right)\partial_z\frac{\mathbf{1}}{1-e^{-2\pi(z-i\tau')/\beta}}+\cdots\,, \end{aligned} \tag{3.11}$$

where dots stand for less singular terms. We further differentiate this term with respect to $\bar z$ using the identity (A.9)

$$\bar\partial\frac{1}{1-e^{-2\pi z/\beta}} = \left(\frac{\beta}{2\pi}\right)\frac{\pi}{2}\delta^{(2)}(z)$$

that is derived in appendix A. We obtain thus

$$\bar\partial T_{zz}(z)\left(-\frac{1}{8\pi}\right):\phi^2:(\tau') = \frac{1}{8\pi}\partial_z\delta^{(2)}(z-i\tau')\mathbf{1}+\cdots \tag{3.12}$$

[6] This should be considered as part of the particular renormalization scheme we choose.

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and the analogous complex conjugated identity.

Substituting (3.12) and the complex conjugated expression into the bulk part of (2.11) we obtain the OPE[7]

$$\Theta_{\rm bulk}(z,\bar z)\phi_u(\tau') = \Theta_{\rm bulk}(z,\bar z)\left(-\frac{1}{8\pi}\right):\phi^2(\tau'): = -\frac{1}{2\pi}\delta^{(2)}(z-i\tau')\mathbf{1}\,. \qquad (3.13)$$

Using this contact term and formulas (3.10), (3.9), (3.4) we find that the identities (2.12) and (2.13) hold.

4. Ising model at criticality with a boundary magnetic field

4.1 The model

A more physical and also more illustrative example of a boundary flow is a 2D Ising model at criticality perturbed by a boundary magnetic field. The system flows from a free spin boundary condition in the UV to the fixed spin boundary condition in the IR. On an infinite half-cylinder the action functional of the model reads [11]

$$S = \frac{1}{2\pi}\int\!\!\int dx d\tau\,(\psi\bar\partial\psi + \bar\psi\partial\bar\psi) + \int d\tau\left(\frac{i}{4\pi}\psi\bar\psi + \frac{1}{2}a\dot a + ih\mu^{1/2}a\,(\omega\psi+\bar\omega\bar\psi)\right). \qquad (4.1)$$

Here $\omega = e^{i\pi/4}$, $\bar\omega = e^{-i\pi/4}$, the complex coordinate is $z = x+i\tau$. The boundary fermion $a(\tau)$ accounts for the double degeneracy of the ground state. Its free propagator reads

$$\langle a(\tau)a(\tau')\rangle_{\rm free} = \frac{1}{2}{\rm sign}(\tau-\tau')\,.$$

The boundary coupling constant h is dimensionless. The equations of motion and the boundary conditions read

$$\bar\partial\psi = 0\,, \qquad \partial\bar\psi = 0 \qquad (4.2)$$

$$\dot a = -ih\mu^{1/2}(\omega\psi+\bar\omega\bar\psi)\,, \qquad (4.3)$$

$$\omega\psi - \bar\omega\bar\psi = -4\pi h\mu^{1/2}a\,. \qquad (4.4)$$

Functionally integrating out the boundary fermion a we obtain a functional integral over the ψ, $\bar\psi$ fields of the form

$$\int D[a]D[\psi,\bar\psi]e^{-S_{\rm bulk}-S_{\partial}} =$$

$$= \int D[\psi,\bar\psi]e^{-S_{\rm bulk}}\exp\left(\frac{h^2\mu}{4}\int\!\!\int d\tau d\tau'\,\Psi(\tau){\rm sgn}(\tau-\tau')\Psi(\tau')\right) \qquad (4.5)$$

where

$$\Psi \equiv \omega\psi + \bar\omega\bar\psi\,.$$

[7]Note that the less singular terms in (3.12) proportional to the identity operator drop out when we take the difference $\bar\partial T_{zz}\phi_u - \partial T_{\bar z\bar z}\phi_u$ while the terms proportional to $\partial_\tau\phi_u$ coming from single contraction terms in (3.11) combine with analogous term in the boundary-boundary OPE in (2.11) to yield the right hand side.

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The boundary term in (4.5) imposes a local boundary condition on the remaining fermionic fields

$$(\partial_\tau - i\lambda)\omega\psi = (\partial_\tau + i\lambda)\bar{\omega}\bar{\psi}\,, \tag{4.6}$$

where

$$\lambda = 4\pi h^2 \mu\,.$$

Note that on a finite length cylinder two sectors: Neveu-Schwarz and Ramond are present in the bulk theory. In the infinite cylinder length limit $L \to \infty$ the Ramond sector states are suppressed by a factor $exp(-L/16\beta)$ due to the Ramond vacuum weight of $1/16$ and thus these states do not survive in the thermodynamic limit we are considering. We will thus consider below only the Neveu-Schwarz sector.

It is easy to find the two-point free fermion Green's functions satisfying the boundary condition (4.6)

$$\begin{aligned}
\langle\psi(z_1)\psi(z_2)\rangle &= \frac{\pi/\beta}{\sinh\left(\frac{\pi}{\beta}(z_1 - z_2)\right)}\,, \\
\langle\psi(z_1)\bar{\psi}(\bar{z}_2)\rangle &= \frac{i\pi/\beta}{\sinh\left(\frac{\pi}{\beta}(z_1 + \bar{z}_2)\right)} + g(z_1 + \bar{z}_2)\,,
\end{aligned} \tag{4.7}$$

where

$$g(z) = -\sum_{n=0}^{\infty} \frac{2i\lambda}{n + 1/2 + \beta\lambda/2\pi} e^{-\frac{2\pi z}{\beta}(n+1/2)}\,. \tag{4.8}$$

4.2 The operator θ

The connected part of the boundary operator $\theta(\tau)$ can be found from conservation equation (2.2). We have

$$T_{zz}(z) = -\frac{1}{2\pi}T(z)\,, \qquad T(z) = -\frac{\mu^{-2}}{2}\psi\partial\psi\,,$$

where the unusual factor of μ^{-2} is due to our scaling conventions. Thus on the boundary we have

$$T_{x\tau}(0,\tau) = i(T_{zz} - T_{\bar{z}\bar{z}})|_{x=0} = \frac{\mu^{-2}}{2\pi}(\psi\partial_\tau\psi + \bar{\psi}\partial_\tau\psi)\,. \tag{4.9}$$

The boundary condition (4.6) then implies

$$\partial_\tau\psi = i\lambda\psi - i(\partial_\tau + i\lambda)\bar{\psi}\,, \tag{4.10}$$

$$\partial_\tau\bar{\psi} = i\lambda\bar{\psi} + i(\partial_\tau - i\lambda)\psi\,. \tag{4.11}$$

Using these formulas we obtain

$$T_{x\tau} = -\frac{i}{4\pi}\partial_\tau(\psi\bar{\psi}) \tag{4.12}$$

and therefore

$$\theta_c = \frac{i\mu^{-1}}{4\pi} :\psi\bar{\psi}:\,. \tag{4.13}$$

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We will derive shortly the term in θ proportional to the identity operator.

Another way to find θ_c is by applying Noether's theorem to the lagrangian (4.1). Taking the classical mass dimension of ψ to be 1/2 we have

$$\theta_c = -\frac{h\mu^{-1}}{2} ia(\omega\psi + \bar{\omega}\bar{\psi}) \,.$$

On the boundary condition (4.6) this expression coincides with (4.13). The last expression however manifestly has the form $\beta_h \phi_h(\tau)$ with $\beta_h = h/2$ and

$$\phi_h = -ia\mu^{-1}(\omega\psi + \bar{\omega}\bar{\psi}) = \frac{i}{2\pi\mu h} :\psi\bar{\psi}: \,.$$

4.3 The partition function

The disk partition function for the model at hand was found in (A.7) using the boundary state formalism. To fix the nonuniversal term present in the partition function in a particular subtraction scheme (to be below) we derive the partition function using the correlators (4.7). It follows from (4.5) that up to a normalization constant the disk partition function can be extracted from the equation

$$\frac{\partial \ln z(h^2\mu)}{\partial h^2\mu} = \frac{1}{4}\iint d\tau d\tau' \langle \Psi(\tau)\Psi(\tau')\rangle \mathrm{sgn}(\tau - \tau') \,. \tag{4.14}$$

Using (4.7) the two-point function in the last expression can be computed to be

$$\begin{aligned} \langle \Psi(\tau)\Psi(\tau')\rangle &= i[\langle\psi(\tau)\psi(\tau')\rangle - \langle\bar{\psi}(\tau)\bar{\psi}(\tau')\rangle] + \langle\psi(\tau)\bar{\psi}(\tau')\rangle + \langle\bar{\psi}(\tau)\psi(\tau')\rangle \\ &= \frac{4\pi/\beta}{\sin\left[\frac{\pi}{\beta}(\tau-\tau')\right]} - \sum_{n=0}^{\infty} \frac{4\lambda}{n + 1/2 + \beta\lambda/2\pi} \sin\left[\frac{2\pi}{\beta}\left(n+\frac{1}{2}\right)(\tau - \tau')\right] . \end{aligned} \tag{4.15}$$

The contribution of the series to (4.14) can be expressed via Euler's psi function while the first term contributes an integral

$$I \equiv \frac{\pi}{\beta}\int_0^\beta d\tau \int_0^\beta d\tau' \frac{1}{|\sin\left[\frac{\pi}{\beta}(\tau-\tau')\right]|} \tag{4.16}$$

that is logarithmically divergent. As shown in appendix A (formula (A.15)) the point splitting regularization plus minimal subtraction yield a renormalized value

$$I_{\mathrm{ren}} = \beta\left(2\ln(\mu\beta) - \psi\left(\frac{1}{2}\right)\right) . \tag{4.17}$$

Collecting all terms we obtain

$$\begin{aligned} \frac{\partial \ln z(h^2)}{\partial h^2} &= \beta\left(2\ln(\mu\beta) - \psi\left(\frac{1}{2}\right)\right) + \sum_{n=0}^{\infty}\frac{4h^2\beta^2\mu}{(n+1/2+2\beta h^2\mu)(n+1/2)} \\ &= \beta 2\ln(\beta\mu) + \beta\psi\left(\frac{1}{2}\right) - \psi\left(2\beta h^2\mu + \frac{1}{2}\right) . \end{aligned} \tag{4.18}$$

This yields

$$z = \frac{C}{\Gamma(\alpha+1/2)} e^{\alpha(\ln(\beta\mu)+\frac{1}{2}\psi(1/2))} , \tag{4.19}$$

where

$$\alpha = 2\beta\mu h^2 ,$$

C is a normalization constant. Up to this constant and nonuniversal terms of the form $e^{\alpha C'}$ that depend on the subtraction scheme the above value of z coincides with the one computed in [13]. The constant C can be fixed by computing a partition function on a finite cylinder that can be canonically normalized and by studying its factorization in the infinite cylinder limit. This yields $C = \sqrt{\pi}$ [13].

The logarithm of partition function (4.19) satisfies the RG equation

$$\mu \frac{\partial \ln z}{\partial \mu} = \frac{h}{2} \frac{\partial \ln z}{\partial h} + 2\beta\mu h^2 . \tag{4.20}$$

Thus the complete θ reads

$$\theta = \frac{i}{4\pi} \mu^{-1} :\psi\bar{\psi}: + 2\beta\mu h^2 \mathbf{1} . \tag{4.21}$$

From (4.19) one readily computes the boundary entropy

$$s(\alpha) = \ln\sqrt{\pi} - \ln\Gamma\left(\alpha + \frac{1}{2}\right) + \alpha\psi\left(\alpha+\frac{1}{2}\right) - \alpha . \tag{4.22}$$

One finds then that $s_{UV} = s(0) = 0$, $s_{IR} = s(\infty) = -\frac{1}{2}\ln 2$ in accordance with the boundary entropy values for free and fixed boundary conditions in the Ising model [10].

4.4 $\langle \phi_h(\tau)\theta(\tau')\rangle_c$ correlator. A contact term.

We are interested in this section in the correlator $\langle \phi_h(\tau)\theta(\tau')\rangle_c$ and particularly in the contact term it contains. Up to a constant factor this correlator equals

$$\langle :\psi\bar{\psi}: (\tau) :\psi\bar{\psi}: (\tau')\rangle_c \equiv G(\tau-\tau') . \tag{4.23}$$

Let us write

$$\langle \psi(\tau)\bar{\psi}(\tau')\rangle = \frac{\pi/\beta}{\sin\left[\frac{\pi}{\beta}(\tau-\tau')\right]} + g[i(\tau-\tau')] \tag{4.24}$$

where the function g is defined in (4.8). Then by Wick's theorem

$$G(\tau-\tau') = (g[i(\tau-\tau')] - g[i(\tau'-\tau)]) \frac{\pi/\beta}{\sin\left[\frac{\pi}{\beta}(\tau-\tau')\right]} - g[i(\tau-\tau')]g[i(\tau'-\tau)] .$$

It is easy to check that the second term is an integrable function while the first term contains a singularity of the order $|\tau-\tau'|^{-1}$. This is unlike in the boundary mass model where the correlator $\langle\theta(\tau)\phi_u(\tau')\rangle_c$ was integrable and there was no need for a contact term that would ensure the integrability. After some work we can single out the singular piece in a simple form and rewrite $G(\tau-\tau')$ as

$$G(\tau-\tau') = -2\lambda\pi \frac{1}{\frac{\beta}{\pi}|\sin(\frac{\pi}{\beta}(\tau-\tau'))|} + f_1(\tau-\tau') + f_2(\tau-\tau') \tag{4.25}$$

where

$$f_1(\tau-\tau') = \frac{2\lambda^2}{\sin(\frac{\pi}{\beta}(\tau-\tau'))}\sum_{n=0}^{\infty}\frac{\sin\left[\frac{\pi}{\beta}(2n+1)(\tau-\tau')\right]}{(n+1/2+\alpha)(n+1/2)}, \tag{4.26}$$

$$f_2(\tau-\tau') = 4\lambda^2\sum_{n,m=0}^{\infty}\frac{\cos\left[\frac{2\pi}{\beta}(\tau-\tau')(m-n)\right]}{(n+1/2+\alpha)(m+1/2+\alpha)}. \tag{4.27}$$

Both functions f_1 and f_2 are integrable. Applying a point splitting plus minimal subtraction to the singular piece (see formula (A.15) in appendix A) we obtain a distribution that we denote

$$R\frac{1}{\frac{\beta}{\pi}|\sin\left(\frac{\pi}{\beta}(\tau-\tau')\right)|}$$

times a factor $-2\lambda\pi$. This distribution acts on a test function $\phi(\tau)$ periodic with period β as

$$\left(R\frac{1}{\frac{\beta}{\pi}|\sin(\frac{\pi}{\beta}(\tau-\tau'))|},\phi\right) :=$$

$$:= \int_0^{\beta/2} d\tau\frac{\phi(\tau)+\phi(\beta-\tau)-2\phi(0)}{\frac{\beta}{\pi}\sin(\frac{\pi}{\beta}(\tau-\tau'))} + \phi(0)\left(2\ln(\mu\beta)-\psi\left(\frac{1}{2}\right)\right). \tag{4.28}$$

Thus we promoted the correlation function $G(\tau-\tau')$ defined for finite separations in (4.25), (4.26), (4.27) to a distribution

$$G(\tau-\tau') = -(2\lambda\pi)R\frac{1}{\frac{\beta}{\pi}|\sin(\frac{\pi}{\beta}(\tau-\tau'))|} + f_1(\tau-\tau') + f_2(\tau-\tau'). \tag{4.29}$$

Now we can compute the integrals involving the distribution (4.29). A straightforward computation yields

$$\int_0^\beta\int_0^\beta d\tau d\tau' G(\tau-\tau') = 16\pi^2\left(\frac{\alpha}{2}\psi\left(\alpha+\frac{1}{2}\right)-\frac{\alpha}{4}\psi(\frac{1}{2})-\frac{\alpha}{2}\ln(\beta\mu)+\alpha^2\psi'\left(\alpha+\frac{1}{2}\right)\right). \tag{4.30}$$

$$\int_0^\beta\int_0^\beta d\tau d\tau' G(\tau-\tau')\cos\left[\frac{2\pi}{\beta}(\tau-\tau')\right] =$$

$$= 16\pi^2\alpha\left(\frac{1}{2}\psi\left(\alpha+\frac{1}{2}\right)-\frac{1}{4}\psi\left(\frac{1}{2}\right)-\frac{1}{2}\ln(\beta\mu)+1\right). \tag{4.31}$$

Note that the renormalized value of the integral (4.30) coincides with the value of

$$(-2\pi i)^2 h^2\frac{\partial^2\ln z}{\partial h\partial h}.$$

This of course was to be expected because we used the same point splitting + minimal subtraction scheme when evaluating both quantities.

4.5 The gradient formula

We would like to check now the gradient formula

$$-\frac{\partial s}{\partial h}=\mu^2\int_0^\beta\!\!\int_0^\beta d\tau d\tau'\langle\phi_h(\tau)\theta(\tau')\rangle_c\left(1-\cos\left[\frac{2\pi}{\beta}(\tau-\tau')\right]\right). \tag{4.32}$$

The RG equation implies that

$$\frac{\partial s}{\partial h}=\frac{2}{h}\mu\frac{\partial s}{\partial\mu}=-\frac{2\mu^2}{h}\frac{\partial^2\ln z}{\partial\mu\partial\mu}.$$

In the last subsection we computed

$$\langle\phi_h(\tau)\theta(\tau')\rangle_c=-\frac{2\mu^{-1}}{h}\frac{1}{16\pi^2}\langle:\psi\bar{\psi}:(\tau):\psi\bar{\psi}:(\tau')\rangle_c\equiv-\frac{2\mu^{-2}}{h}\frac{1}{16\pi^2}G(\tau-\tau'),$$

where G is a distribution given in (4.29). Checking the gradient formula (4.32) then boils down to checking

$$\mu^2\frac{\partial^2\ln z}{\partial\mu^2}=-\frac{1}{16\pi^2}\int_0^\beta\!\!\int_0^\beta d\tau d\tau' G(\tau-\tau')\left(1-\cos\left[\frac{2\pi}{\beta}(\tau-\tau')\right]\right). \tag{4.33}$$

The left hand side is immediately computed using (4.19) with the result

$$\mu^2\frac{\partial^2\ln z}{\partial\mu^2}=\alpha-\alpha^2\psi'\left(\alpha+\frac{1}{2}\right)$$

that indeed equals to the difference of integrals (4.30) and (4.31) times a factor of $-(16\pi^2)^{-1}$.

4.6 Contact term in the OPE of ϕ_h with $\Theta_{\rm bulk}$

As in the bosonic model we can derive the contact term in the OPE of $\Theta_{\rm bulk}$ with ϕ_h by using Ward identity (2.11) and the OPE's of ϕ_h with $T(z)$ and $\bar{T}(\bar{z})$. The term proportional to the identity operator in the OPE of $T(z)$ with $:\psi\bar{\psi}:(\tau')$ reads

$$:\psi\partial\psi:(z):\psi\bar{\psi}:(\tau')=\mathbf{1}[\langle\psi(z)\bar{\psi}(\tau')\rangle\partial_z\langle\psi(z)\psi(\tau')\rangle-\langle\psi(z)\psi(\tau')\partial_z\langle\psi(z)\bar{\psi}(\tau')\rangle]+\cdots.$$

The most singular terms of the order $(z-i\tau)^{-3}$ in the above expression cancel out and the leading singularity comes from the logarithmic subdivergence in $\langle\psi\bar{\psi}\rangle$. Up to less singular terms we have

$$:\psi\partial\psi:(z):\psi\bar{\psi}:(\tau')=\mathbf{1}\left(\frac{2\pi}{\beta}\right)\partial_z\frac{(g(z-i\tau')+4i\lambda)}{1-e^{-2\pi(z-i\tau')}}+\cdots. \tag{4.34}$$

As $z\to i\tau$ the function $g(z-i\tau')$ has asymptotics of the form

$$g(z-i\tau')=2i\lambda\ln(1-e^{-2\pi(z-i\tau')/\beta})+2i\lambda\left[\gamma+\psi\left(\alpha+\frac{1}{2}\right)\right]+\mathcal{O}(z-i\tau')$$

where is the Euler's constant. Differentiating the above OPE with respect to $\bar{z}$ in the distributional sense we find using formulas (A.9), (A.16), (B.4) that

$$\bar{\partial}_{\bar{z}}\Big[:\psi\partial\psi:(z):\psi\bar{\psi}:(\tau')\Big] = \\ = \mathbf{1}(2\pi i\lambda)\left[\frac{1}{2}\psi\left(\alpha+\frac{1}{2}\right)-\frac{1}{4}\psi\left(\frac{1}{2}\right)-\frac{1}{2}\ln(\mu\beta)+1\right]\partial_z\delta^{(2)}(z-i\tau')+\ldots \quad (4.35)$$

where the dots stand for less singular terms proportional to $\delta(z-i\tau')\partial\phi_h$. We also have an analogous contact term with ∂T. Using these contact terms and the Ward identity (2.11) we obtain a contact term with $\Theta_{\rm bulk}$ of the form

$$\Theta_{\rm bulk}(x,\tau):\psi\bar{\psi}:(\tau') = \\ \mathbf{1}(-8i\pi h^2)\left[\frac{1}{2}\psi\left(\alpha+\frac{1}{2}\right)-\frac{1}{4}\psi(1/2)-\frac{1}{2}\ln(\mu\beta)+1\right]\delta(x)\delta(\tau-\tau')\,. \quad (4.36)$$

Using this expression along with (4.30), (4.31) we find that equations (2.12), (2.13) indeed hold.

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A. Some distributions supported at a point on the boundary

In this appendix we discuss how one can define distributions formally expressed as $\bar{\partial}f(z)$ where $f(z)$ is a function holomorphic in the $x>0$ region of the cylinder and contains a singular point on the boundary that without loss of generality can be chosen to be $x=0,\tau=0$. We further assume that $f(z)$ diverges at that point slower than z^{-2}. In particular we will be interested in defining

$$\bar{\partial}\frac{1}{1-e^{-2\pi z/\beta}} \quad (A.1)$$

and

$$\bar{\partial}\,\frac{\ln(1-e^{-2\pi z/\beta})}{1-e^{-2\pi z/\beta}}\,. \quad (A.2)$$

It is intuitively clear that any distribution expressed as $\bar{\partial}f$ where f satisfies the above conditions should be supported at the singularity point $z=0$. By dimensional reasons the answer should be proportional to $\delta^{(2)}(z)$. To obtain a rigorous definition one can proceed as follows. Let $\varphi(z,\bar{z})$ be a test function which is smooth and fast decreasing at infinity. Consider a formal identity

$$\int_0^\infty dx\int_0^\beta d\tau\,\varphi\bar{\partial}f = -\int_0^\infty dx\int_0^\beta d\tau\,\varphi\bar{\partial}f+\frac{1}{2}\int_0^\beta d\tau\,\varphi f|_{x=\infty}-\frac{1}{2}\int_0^\beta d\tau\,\varphi f|_{x=0} \quad (A.3)$$

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that follows from a formal application of Stokes formula. The first term on the right hand side is well defined as the singularity of f is integrable in two-dimensions. The second term in the above equation vanishes because the test function is fast decreasing at infinity. In the last term one has to regularize a singular function of one variable $f(x=0,\tau)$ and promote it to a one-dimensional distribution. Thus we see that the problem of defining a distribution $\bar{\partial} f$ on a space with boundary (a half-cylinder in our case) boils down to defining a one-dimensional distribution to be denoted $Rf(\tau)$ that regularizes the restriction of f to the boundary. This can be done in a standard fashion, see e.g. [14]. As we stick to point splitting regularization throughout the paper we will define a distribution $Rf(\tau)$ by the formula

$$\int_0^\beta d\tau \varphi R f := \lim_{\epsilon\to 0}\left[\int_{0+\epsilon}^{\beta-\epsilon} d\tau \varphi f|_{x=0} - C(\epsilon)\varphi(0)\right], \tag{A.4}$$

where $C(\epsilon)$ is a counterterm ensuring the convergence of the whole expression. In the case when no counterterm is needed and it can thus be set to zero the right hand side gives the principal value of the corresponding integral. When $C(\epsilon)$ is present it is defined up to an essential ambiguity $C(\epsilon) \to C(\epsilon) + Const.$

As the first term in (A.3) is well defined we can rewrite it using Stokes formula as

$$\int_0^\infty dx \int_0^\beta d\tau\, \varphi\bar{\partial} f = \lim_{\epsilon\to 0}\iint_{|z|\geq\epsilon} dx d\tau\, \varphi\bar{\partial} f = -\frac{1}{2}\lim_{\epsilon\to 0}\left[\int_{0+\epsilon}^{\beta-\epsilon} d\tau\varphi f|_{x=0} + \int_{|z|=\epsilon, x\geq 0} dz\, \varphi f\right].$$

Substituting the last expression and expression (A.4) into (A.3) we obtain

$$\int_0^\infty dx \int_0^\beta d\tau\, \varphi\bar{\partial} f := \frac{1}{2}\lim_{\epsilon\to 0}\left[C(\epsilon)\varphi(0) + \int_{|z|=\epsilon, x\geq 0} dz\, \varphi f\right]. \tag{A.5}$$

This expression makes it manifest that

$$\bar{\partial} f = C_f\, \delta^{(2)}(z) \tag{A.6}$$

and that up to the already noted ambiguity in $C(\epsilon)$ the coefficient C_f depends only on the behavior of f in the vicinity of $z=0$. For practical purposes however it is convenient to use a different representation of the constant C_f. By applying Cauchy formula to the function f in region $|z|\geq\epsilon$ we have

$$\int_{|z|=\epsilon, x\geq 0} dz\, f = -\int_{\epsilon}^{\beta-\epsilon} d\tau f|_{x=0} + \int_0^\beta d\tau\, f|_{x=\infty}.$$

Using this formula we deduce from (A.5) and (A.4) that

$$C_f = -\frac{1}{2}\int_0^\beta d\tau R f + \frac{1}{2}\int_0^\beta d\tau\, f|_{x=\infty}. \tag{A.7}$$

Let us apply now this formula to functions (A.1), (A.2). The restriction of function f in (A.1) to the boundary reads

$$f_1(\tau) = \frac{1}{1 - e^{-2\pi i\tau/\beta}} = \frac{1}{2} - \frac{i}{2}\cot\left(\frac{\pi\tau}{\beta}\right). \tag{A.8}$$

It is clear from this expression that the principal value integral of $f_1(\tau)$ exists. We thus set $Rf_1 = \text{P.V.}f_1$. Taking into account a nontrivial contribution at infinity one readily obtains from (A.7)

$$\bar{\partial}\frac{1}{1 - e^{-2\pi z/\beta}} = \left(\frac{\beta}{2\pi}\right)\frac{\pi}{2}\delta^{(2)}(z). \tag{A.9}$$

This formula is to be compared with standard formula derived in the absence of boundary when $z = z'$ is a point in the bulk:

$$\bar{\partial}_{\bar{z}}\frac{1}{z - z'} = \pi\delta^{(2)}(z - z').$$

We see that in our regularization scheme (A.9) corresponds to having a half of delta function in the case when the singularity point is on the boundary.

To derive a distribution corresponding to (A.2) we first note the formula

$$\ln(1 - e^{-2\pi i\tau/\beta}) = \ln\left(2\left|\sin\left(\pi\frac{\tau}{\beta}\right)\right|\right) + \frac{i}{2}\left(\pi - 2\pi\frac{\tau}{\beta}\right) \tag{A.10}$$

where it is assumed that $0 \le \tau \le \beta$. We further notice that the function f entering (A.2) vanishes at infinity. Thus to find the corresponding C_f it suffices to define a regularized integral

$$C_f = -\frac{1}{2}\int_0^\beta d\tau\, R\frac{\ln(1 - e^{-2\pi i\tau/\beta})}{1 - e^{-2\pi i\tau/\beta}}.$$

Multiplying (A.8) by (A.10) and dropping the terms vanishing under the principal value integration we arrive at the expression

$$C_f = -\frac{1}{4}\left[\int_0^\beta d\tau\,\ln\left(2\left|\sin\left(\pi\frac{\tau}{\beta}\right)\right|\right) - \frac{1}{2}\int_0^\beta d\tau\, R\cot\left(\frac{\pi\tau}{\beta}\right)\left(2\frac{\pi\tau}{\beta} - \pi\right)\right].$$

Keeping the limits of integration regularized we can rearrange the terms separating a finite and a singular piece in this integral as

$$\int_\epsilon^{\beta/2} d\tau\left[-\frac{1}{2}\ln\left(2\left|\sin\left(\frac{\pi\tau}{\beta}\right)\right|\right) + \frac{\pi\tau}{2\beta}\cot(\pi\tau/\beta) + \left(\frac{\pi}{4}\right)\frac{\cos(\pi\tau/\beta) - 1}{\sin(\pi\tau/\beta)}\right] -$$
$$-\frac{\pi}{4}\int_\epsilon^{\beta/2}\frac{d\tau}{\sin(\pi\tau/\beta)}. \tag{A.11}$$

The integral on the first line converges in the limit $\epsilon \to 0$ and can be evaluated using formula (B.2). Namely the value of the convergent term in (A.11) equals $(\beta/2)\ln 2$. We can extract a finite value out of the divergent term

$$\int_\epsilon^{\beta/2} \frac{d\tau}{\sin(\pi\tau/\beta)} \tag{A.12}$$

by adding a minimal counterterm:

$$-C(\epsilon) = \frac{\beta}{\pi}(\ln(\epsilon) + \ln(\mu\beta))\,,$$

where μ is the renormalization scale. The remaining finite piece can be computed using the following trick. Consider an integral

$$I_2(\nu) \equiv \int_0^\pi d\theta \left[\sin^2\left(\frac{\theta}{2}\right)\right]^\nu = \frac{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})}{\Gamma(1+\nu)}\,, \quad \nu > -\frac{1}{2}\,. \tag{A.13}$$

The minimally subtracted value of this integral at $\nu = -1/2$ can be obtained by taking $\nu = -1/2 + \epsilon$, $\epsilon > 0$, expanding in powers of ϵ and subtracting the pole. This yields the value

$$I_2\left(-\frac{1}{2}\right) = -\psi\left(\frac{1}{2}\right),$$

where ψ is the logarithmic derivative of the Euler's gamma function. This fixes the finite part remaining after the minimal subtraction in the divergent term (A.12). We have

$$\int_0^{\beta/2} d\tau\, R\frac{1}{\sin(\pi\tau/\beta)} = \lim_{\epsilon\to 0}\left[\int_\epsilon^{\beta/2} \frac{d\tau}{\sin(\pi\tau/\beta)} - C(\epsilon)\right] = \left(\frac{\beta}{2\pi}\right)\left[2\ln(\mu\beta) - \psi\left(\frac{1}{2}\right)\right]. \tag{A.14}$$

In a similar fashion a minimally subtracted value of the following integral that is used in the main body of the paper is obtained

$$I_{ren} \equiv \frac{1}{\beta}\int_0^\beta d\tau \int_0^\beta d\tau' R \frac{1}{|\sin\left[\frac{\pi}{\beta}(\tau-\tau')\right]|} = \left(\frac{\beta}{\pi}\right)\left[2\ln(\mu\beta) - \psi\left(\frac{1}{2}\right)\right]. \tag{A.15}$$

Collecting all pieces together in (A.11) we finally obtain

$$\bar\partial \frac{\ln(1-e^{-2\pi z/\beta})}{1-e^{-2\pi z/\beta}} = \left(\frac{\beta}{2\pi}\right)\pi\left[\ln 2 - \frac{1}{2}\ln(\mu\beta) + \frac{1}{4}\psi\left(\frac{1}{2}\right)\right]\delta^{(2)}(z)\,. \tag{A.16}$$

Note also that applying the general formula (A.7) we find

$$\bar\partial \ln(1-e^{-2\pi z/\beta}) = 0$$

that agrees with the naive dimensional intuition.

B. Some useful integrals and series

$$\int_0^{2\pi} d\theta \, \frac{\sin(\theta(n+1/2))}{\sin(\theta/2)} = \pi \tag{B.1}$$

$$\int_0^{\pi} dx \, \ln(\sin(x)) = -\pi \ln 2 \,. \tag{B.2}$$

$$\psi(x) = -\gamma + (x-1) \sum_{n=1}^{\infty} \frac{1}{n(x+n-1)} \,. \tag{B.3}$$

$$\psi\left(\frac{1}{2}\right) = -\gamma - 2\ln 2 \,, \tag{B.4}$$

$$\sum_{n=0}^{\infty} \frac{\sin((2n+1)\tau)}{n+1/2} = \frac{\pi}{2} \;\text{ for }\; 0 < \tau < \pi \,, \quad -\frac{\pi}{2} \;\text{ for }\; \pi < \tau < 2\pi \,. \tag{B.5}$$

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